Programmable quantum-state discriminator by nuclear magnetic resonance

T. Gopinath, Ranabir Das, and Anil Kumar

NMR Quantum Computing and Quantum Information Group, Department of Physics, and Sophisticated Instruments Facility, Indian Institute of Science, Bangalore 560012, India
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A programmable quantum-state discriminator is implemented by using nuclear magnetic resonance. We use a two-qubit spin-1/2 system, one for the data qubit and one for the ancilla (program) qubit. This device does the unambiguous (error-free) discrimination of a pair of states of the data qubit that are symmetrically located about a fixed state. The device is used to discriminate both linearly polarized states and elliptically polarized states. The maximum probability of successful discrimination is achieved by suitably preparing the ancilla qubit. It is also shown that the probability of discrimination depends on the angle of the unitary operator of the protocol and ellipticity of the data qubit state.

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I. INTRODUCTION

Researchers have studied the possibility of performing computations using quantum systems and conjectured that a machine based on quantum mechanical principles might be able to solve certain types of problems more efficiently than can be done on conventional computers [1–3]. Later Lloyd proposed that such a quantum computer might be built from an array of coupled two-state quantum systems [4]. Its theoretical possibility has generated a lot of enthusiasm for its experimental realization [5–10]. In parallel with quantum computation, the related field of quantum-information theory is developed, which forms the quantum analog of classical-information theory [11]. Several techniques are being exploited for quantum computing and quantum-information processing, including nuclear magnetic resonance [12–19].

Recently quantum-state discrimination has been studied extensively in the context of quantum communication and quantum cryptography [20–27]. Quantum-state discrimination is the problem of determining the quantum state, given the constraint that it belongs to the previously specified set of nonorthogonal states. One of the characteristic features of quantum mechanics is that it is impossible to devise a measurement that can distinguish nonorthogonal states perfectly [10]. However, one can distinguish them with a finite probability by an appropriate measurement strategy. There are two different optimal strategies of discrimination: (i) probabilistic discrimination (conclusive result, but error may appear) and (ii) unambiguous discrimination (inconclusive result may appear, but no error). The unambiguous discrimination of two pure states was investigated by Ivanovic [26], and the optimal procedure was given by Peres [27]. CheffeIs and Barnett have generalized Peres’s solution to an arbitrary number of equally probable states which are related by a symmetry transformation [28]. The first experiment to discriminate two nonorthogonally polarized single-photon states of light was done by Huttner et al. [29].

Quantum measurement is the final step of any quantum computation. In many situations the choice of an optimal measurement depends on the task to be performed. A quantum multimeter is a quantum-measurement device which can perform a specific class of generalized measurements in such a way that each member of this class is selected by a particular quantum state of a program register [30–34]. The parameters determining the character of quantum measurement can be encoded in a quantum state of a program register [35–37]. Dusek and Buzek [30] have shown that a pair of nonorthogonal states of a qubit can be discriminated and the measurement to be done for this discrimination is decided by the state of the program qubit (ancilla qubit). One can discriminate several pairs of states of a qubit by using the same protocol [30]. Such a quantum device is known as a quantum multimeter for the discrimination of a pair of qubit states. Recently Soubusta et al. [38] have also demonstrated experimentally the possibility of controlling the discrimination process by the quantum state of an ancilla qubit, in linear optics by performing the partial measurement in the Bell basis. Cryptographic applications of quantum-state discrimination have been extensively studied in the literature [39–42].

Nuclear magnetic resonance (NMR) has played a leading role for the practical demonstration of quantum algorithms and gates [12–19]. The unitary operators needed for implementation of these quantum circuits have mostly been realized using spin-selective as well as transition-selective radio-frequency pulses and coupling evolution, utilizing spin-spin (J) or dipolar couplings among the spins [12–19]. In this paper we demonstrate the implementation of a quantum-state discriminator which discriminates the pair of nonorthogonal states as well as orthogonal states which are symmetric about a particular state, conditioned on the state of the ancilla qubit. We use spin-selective pulses and evolution under J coupling for the implementation. Projective measurement required for the discrimination is simulated by a method given by Collins [43]. Our experimental results are in agreement with the theoretical results [30]. To the best of our knowledge this is the first experimental demonstration of programmable quantum-state discriminator by NMR.

In Sec. II, we discuss the theory of discrimination of both elliptically and linearly polarized states. Experimental details and results of different experiments are given in Sec. III. Results are concluded in Sec. IV. In the Appendix, unitary operators of ideal pulses are derived.
states and later the linearly polarized states. We first describe the discrimination of elliptically polarized states. A quantum circuit for the discrimination is shown in Fig. 1.

Elliptically polarized states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the data qubit. They are symmetrically placed with respect to $|0\rangle$. When $2\theta_1=90^\circ$ the two states are orthogonal. Ellipticity $e$ is defined as $e=y/x$. When $y=0$, $e=0^\circ$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly polarized states. In this paper we experimentally demonstrate the discrimination of both elliptically and linearly polarized states, which are symmetrically placed with respect to $|0\rangle$. In the following protocol discriminates pair of elliptically polarized states of the data qubit unambiguously. The protocol uses one ancilla qubit. One can switch the apparatus to work with several different pairs of data qubit states as shown in (a), then ancilla qubit $|\psi_A\rangle$ is also elliptically polarized state [not shown in (a)].

II. THEORY

The following protocol discriminates pair of elliptically polarized states of the data qubit unambiguously (error free). Let the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the data qubit be [Fig. 1(a)]

$$|\psi_1\rangle = (x \cos \theta_1 + iy \sin \theta_1)|0_D\rangle + (x \sin \theta_1 - iy \cos \theta_1)|1_D\rangle,$$

$$|\psi_2\rangle = (x \cos \theta_1 + iy \sin \theta_1)|0_D\rangle - (x \sin \theta_1 - iy \cos \theta_1)|1_D\rangle.$$  \hspace{1cm} (1)

Ellipticity $(e)$ of the data qubit states is defined as $\tan(e) = y/x$. Here $y=x$ corresponds to circularly polarized states and $y=0$ ($e=0$) corresponds to linearly polarized states [Fig. 1(b)]. The protocol uses one ancilla (program) qubit for the discrimination. A quantum circuit for the discrimination is shown in Fig. 2. In this circuit the first qubit is the data qubit $|\psi_D\rangle$ and the second qubit is the ancilla qubit $|\psi_A\rangle$. The data qubit can be either $|\psi_1\rangle$ or $|\psi_2\rangle$. The aim of the protocol (Fig. 2) is to determine whether the data qubit is $|\psi_1\rangle$ or $|\psi_2\rangle$, knowing the angle between $|\psi_1\rangle$ and $|\psi_2\rangle$. It is shown that a pair of data qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ can be discriminated by suitably preparing the ancilla qubit. One can switch the apparatus to work with several different pairs of data qubit states. In this paper we experimentally demonstrate the discrimination of both elliptically and linearly polarized states, and compare the results with simulations. In the following, we first describe the discrimination of elliptically polarized states and later the linearly polarized states.

![Fig. 1. (a) Pictorial representation of elliptically polarized states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the data qubit. They are symmetrically placed with respect to $|0\rangle$. When $2\theta_1=90^\circ$ the two states are orthogonal. Ellipticity $e$ is defined as $e=y/x$. When $y=0$, $e=0^\circ$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly polarized states. (b) Pictorial representation of linearly polarized states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the data qubit, $|\psi_A\rangle$ is the ancilla qubit. When the data qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ are elliptically polarized states as shown in (a), then ancilla qubit $|\psi_A\rangle$ is also elliptically polarized state [not shown in (a)].](Image 108x526 to 240x734)

![Fig. 2. Quantum circuit for the discrimination of data qubit state $|\psi_D\rangle=|\psi_1\rangle$ or $|\psi_2\rangle$, using an ancilla qubit, prepared in state $|\psi_A\rangle$. The unitary operator $U$ needed for such a protocol consists of two CNOT gates, two NOT gates (X), and two other single-qubit gates $u_1$ and $u_2$. For projective measurement a controlled-$\sigma_z$ gate is needed at the end of the protocol, as described in the text.](Image 341x656 to 533x734)

Elliptically polarized states $|\psi_1\rangle$ and $|\psi_2\rangle$ [Eq. (1)] can be rewritten as

$$|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle,$$

$$|\psi_2\rangle = a_1|0\rangle - b_1|1\rangle,$$  \hspace{1cm} (2)

where $a_1=x \cos \theta_1 + iy \sin \theta_1$ and $b_1=x \sin \theta_1 - iy \cos \theta_1$ are complex numbers, where by definition $x^2+y^2=1$. $a_1$ and $b_1$ can also be written in polar form as

$$a_1 = e^{i\phi_1} \cos(\eta), \quad b_1 = e^{i\phi_2} \sin(\eta).$$  \hspace{1cm} (3)

where

$$\tan(\eta) = \frac{|b_1|}{|a_1|}, \quad \tan(\phi_1) = \frac{y \sin \theta_1}{x \cos \theta_1},$$

$$\tan(\phi_2) = \frac{-y \cos \theta_1}{x \sin \theta_1}.$$  \hspace{1cm} (4)

Then $|\psi_1\rangle$ and $|\psi_2\rangle$ can be written as general states on the Bloch sphere [10],

$$|\psi_1\rangle = \cos(\eta)|0\rangle + e^{i\phi} \sin(\eta)|1\rangle,$$

$$|\psi_2\rangle = \cos(\eta)|0\rangle - e^{i\phi} \sin(\eta)|1\rangle,$$  \hspace{1cm} (5)

where $\phi=\phi_2-\phi_1$ with the overall phase ($e^{i\phi_1}$) being neglected.

Let the ancilla qubit (program qubit) be

$$|\psi_A\rangle = a_2|0\rangle + b_2|1\rangle,$$  \hspace{1cm} (6)

where $a_2=x \cos \theta_2 + iy \sin \theta_2$ and $b_2=x \sin \theta_2 - iy \cos \theta_2$. To discriminate $|\psi_1\rangle$ and $|\psi_2\rangle$, the condition on $a_2$ and $b_2$ is derived as follows.

The total input state is $|\psi_{DA}\rangle=|\psi_D\rangle \otimes |\psi_A\rangle$, where the data qubit $|\psi_D\rangle$ is either $|\psi_1\rangle$ or $|\psi_2\rangle$ [Eq. (2)];

$$|\psi_{DA}\rangle = (a_1a_2|0_D0_A\rangle + a_1b_2|1_D1_A\rangle)$$

$$\pm (b_1a_2|1_D0_A\rangle + b_1b_2|1_D1_A\rangle).$$  \hspace{1cm} (7)

The sign of the second term in Eq. (7) determines whether the data qubit is $|\psi_1\rangle$ or $|\psi_2\rangle$. The protocol for the discrimination requires a unitary transformation on $|\psi_{DA}\rangle$ given by Ref. [30].
where $\alpha$ is a fixed parameter which does not depend on the data and program qubits states. The unitary operator given in Eq. (8) is a rotation in the subspace spanned by $|0_D0_A\rangle$ and $|0_D1_A\rangle$, which is achieved here by two controlled-NOT gates and four single-qubit gates (Fig. 2), where the single-qubit gates are given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix},$$

$$U_2 = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix},$$

and the controlled-NOT (CNOT) gate is given by

$$G_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (10)$$

After the application of the unitary transformation $U$ [Eq. (8)], the final state is

$$U|\psi_{D\lambda}\rangle = (a_1a_2 \cos \alpha - a_1b_2 \sin \alpha)|0_D0_A\rangle + (a_1a_2 \sin \alpha + a_1b_2 \cos \alpha)|0_D1_A\rangle,$$

and

$$|\psi_{D\lambda}\rangle = \pm [b_1a_2|0_D0_A\rangle + b_1b_2|1_D1_A\rangle]. \quad (11)$$

For successful discrimination of data qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$, the condition on the coefficients of Eq. (11) is

$$(a_1a_2 \cos \alpha - a_1b_2 \sin \alpha) = b_1a_2. \quad (12)$$

This yields

$$U|\psi_{D\lambda}\rangle = \pm [b_1a_2|0_D0_A\rangle + b_1b_2|1_D1_A\rangle]. \quad (13)$$

Equation (13) can be rewritten as

$$U|\psi_{D\lambda}\rangle = \sqrt{2b_1a_2} \pm (a_1a_2 \sin \alpha + a_1b_2 \cos \alpha)|0_D1_A\rangle,$$

where $|\pm\rangle = 1/\sqrt{2}(|0_D0_A\rangle \pm |1_D0_A\rangle).$ Notice that $U|\psi_{D\lambda}\rangle$ contains a term $|\pm\rangle$, then the initial state of the data qubit is $|\psi_2\rangle$. If the initial state contains $|\psi_1\rangle$, then the initial state of the data qubit is $|\psi_1\rangle$. The square of the coefficient $(\sqrt{2b_1a_2})$ of the state $|\pm\rangle$ is called the probability ($P$) of discrimination [30].

Equation (12) gives the condition on the ancilla qubit state. For example, when $\alpha = 90^\circ$, $(b_2/a_2) = -(b_2/a_1)$, and from Eqs. (2) and (6) it is seen that $|\psi_1\rangle = |\psi_2\rangle$. However, for other values of $\alpha$, $|\psi_1\rangle$ differs from $|\psi_1\rangle$ and $|\psi_2\rangle$. Hence $\alpha = 90^\circ$ is a special case of Eq. (12).

In NMR, the measurement is performed on an ensemble and the results are contained in the expectation values $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$ of Pauli spin matrices, which in the frequency space yield intensities of various transitions. From Eq. (14) it is noted that the two transitions of the data qubit have different intensities. The $|0_D0_A\rangle \leftrightarrow |1_D0_A\rangle$ transition has the intensity $\pm 2b_1a_2^2$, and the $|0_D1_A\rangle \leftrightarrow |1_D1_A\rangle$ transition has the intensity $\pm (a_1a_2 \sin \alpha + a_1b_2 \cos \alpha)|p_1\rangle$. To find whether the final state [Eq. (14)] contains $|+\rangle$ or $|\pm\rangle$, one has to do a projective measurement on the state $|\pm\rangle$. To simulate the projective measurement, we use a method given by Collins for an expectation value quantum search [43–45]. In our experiments, the goal of the projective measurement is to collapse the state of the ancilla qubit [given by Eq. (14)] to $|0_A\rangle$ so that the data qubit gives only one peak which is the coherence of the superposition state $|\pm\rangle$ (or $|0_D0_A\rangle \leftrightarrow |1_D0_A\rangle$ transition). One can collapse the state of the ancilla qubit to $|0_A\rangle$ by adding two experiments (measurement on the data qubit), one without a controlled-$\sigma_x$ gate and one with a controlled-$\sigma_x$ gate (Fig. 2). Here the controlled-$\sigma_x$ gate is given by the unitary transformation

$$\sigma_x' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (15)$$

In the first experiment [Eq. (14)], the intensities of the data qubit transitions corresponding to $|0_D0_A\rangle \leftrightarrow |1_D0_A\rangle$ and $|0_D1_A\rangle \leftrightarrow |1_D1_A\rangle$ are $\pm 2b_1a_2^2$ and $\pm (a_1a_2 \sin \alpha + a_1b_2 \cos \alpha)|p_1\rangle$, respectively. Thus when the two experiments are added, the intensity of the $|0_D1_A\rangle \leftrightarrow |1_D1_A\rangle$ transition goes to zero and that of $|0_D0_A\rangle \leftrightarrow |1_D0_A\rangle$ to $\pm 2b_1a_2^2$. Hence by the above procedure the ancilla qubit state is collapsed to $|0_A\rangle$, and the phase of the observed transition yields the result of the measurement. If the phase is positive, then the data qubit is $|\psi_1\rangle$, and if it is negative, then the data qubit is $|\psi_2\rangle$. The resultant intensity $2b_1^2a_2^2$ gives the probability of successful discrimination.

For the case of linearly polarized states [Eq. (2), $\gamma = 0$; $a_1 = \cos \theta_1$, $b_1 = \sin \theta_1$], the data qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ are schematically shown in Fig. 1(b), and $\gamma$ and $\phi$ of Eq. (5) are, respectively, given by $\theta_1$ and zero. In this case ancilla qubit state [Fig. 1(b)] is given by Eq. (6), with $a_1 = \cos \theta_2$ and $b_2 = \sin \theta_2$. The rest of the procedure to discriminate $|\psi_1\rangle$ and $|\psi_2\rangle$ remains the same and the probability of discrimination is given by $2b_1^2a_2^2$.

III. EXPERIMENT

In NMR spin–1/2 nuclei having sufficiently different Larmor frequencies and weakly coupled to each other by indirect exchange ($J$) couplings are used as qubits. The Hamil-
We have used a carbon-13 labeled $^{13}$CHCl$_3$ as a two-qubit system, where the proton ($^1$H) and the labeled carbon ($^{13}$C) act as two individual qubits. $J$-coupling between $^{13}$C and $^1$H is 209 Hz. The measured longitudinal relaxation times of $^1$H and $^{13}$C are: $^1$H ($T_1 = 4.8$ s and $T_2 = 3.3$ s) and $^{13}$C ($T_1 = 17.2$ s and $T_2 = 0.35$ s). To implement the circuit of Fig. 2, the data ($^1$H) and ancilla ($^{13}$C) qubits have to be first prepared in a pure state. However, in NMR pure states are difficult to prepare; instead, we prepare pseudopure states which mimic the pure states. Several methods are known for the preparation of pseudopure states [46–52]. Here we use a spatial averaging method [53] to prepare the pseudopure state using the pulse sequence given in Fig. 3. This pulse sequence [53] is specific to the labeled $^{13}$C–$^1$H system and different from the homonuclear case. The details of the preparation of the pseudopure state are given in the figure captions. The spectra of the equilibrium state and pseudopure state are shown in Fig. 4. After preparation of the pseudopure state, the quantum circuit of Fig. 2 is implemented by the pulse sequence given in Fig. 5. The pulse sequence in Fig. 5 consists of three parts.

(i) Preparation of the initial state ($|\psi_{DA}\rangle$): After preparation of the pseudopure state ($|0_D,0_A\rangle$), the data qubit ($^1$H) is prepared in the elliptically polarized state $|\psi_D\rangle$ [Eq. (5)] by applying a $2\eta$ pulse of appropriate phase on the data qubit state $|0_D\rangle$. To prepare the data qubit in state $|\psi_1\rangle$ or $|\psi_2\rangle$, the phase of the $2\eta$ pulse is $(\pi/2+\phi)$ or $-(\pi/2+\phi)$, respectively (Appendix). The ancilla qubit ($^{13}$C) is prepared in state $|\psi_A\rangle$ [Eq. (6)] by using Eq. (12). For example, for $\alpha=90^\circ$, since $|\psi_A\rangle=|\psi_2\rangle$, the ancilla qubit $|\psi_A\rangle$ is prepared by another $2\eta$ pulse. For arbitrary $\alpha$, the ancilla qubit is prepared using Eq. (12) with appropriate pulse angle and phase. In the case of the linearly polarized state [Eq. (2), $y=0$], $\eta=\theta$, and $\phi=0^\circ$, the data qubit can be prepared in states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively, by applying $(2\theta_1)_z$ and $(2\theta_1)_y$ pulses (Appendix) on $|0_D\rangle$. The ancilla qubit [Eq. (6)] is then prepared by applying a $(2\theta_2)_y$ pulse on $|0_A\rangle$, where $2\theta_2$ is calculated according to Eq. (12). From Eq. (12), $2\theta_2$ can take positive as well as negative values. For example, when $\alpha=60^\circ$ and $2\theta_1=20^\circ$, $40^\circ$, $60^\circ$, $80^\circ$, $90^\circ$, $100^\circ$, $120^\circ$, $140^\circ$, and $160^\circ$, $2\theta_2$ takes the values $41^\circ$, $17.8^\circ$, $-10.2^\circ$, $-42.8^\circ$, $-60^\circ$, $-77.2^\circ$, $-109.8^\circ$, $-137.8^\circ$, and $-161^\circ$, respectively. Here one should note that the $-2\theta_2$ pulse is identical to the $(2\theta_2)_y$ pulse.

(ii) Applying the unitary operator $U$: The unitary operator $U$ (Fig. 2) is prepared by using two CNOT gates, two NOT gates and two other single-qubit gates $u_1$ and $u_2$ (Fig. 5). The NOT gates on the data qubit ($^1$H) are implemented by a $(\pi)_x$ pulse and $u_1$ and $u_2$ on the ancilla qubit by $(\alpha)_{-y}$ and $(\alpha)_y$ pulses, respectively, on $^{13}$C. The CNOT gate is implemented...
The pulse sequence for implementation of the quantum circuit of Fig. 2. The data qubit ($^1$H) is prepared in elliptically polarized states $|\psi_i\rangle$ and $|\psi_f\rangle$ [Eq. (5)] by $(2\eta_1)e^{(2\eta_2+\phi_0)}$ and $(2\eta_2)e^{-(2\eta_2+\phi_0)}$ pulses, respectively, and an ancilla qubit ($^{13}$C) is prepared in state $|\psi_2\rangle$ by $(2\eta_1)e^{(2\eta_2+\phi_0)}$ pulse for $\alpha=90^\circ$. In the case of linearly polarized states [Eq. (2), $\gamma=0^\circ$], the data qubit is prepared in states $|\psi_i\rangle$ and $|\psi_f\rangle$ by $(2\theta_1)_{\pi/2}$ and $(2\theta_2)_{\pi/2}$ pulses, respectively, and the ancilla qubit is prepared in state $|\psi_2\rangle$ by $(2\theta_2)_{\pi/2}$ pulse, where $2\theta_2$ is calculated according to Eq. (12). Figure 2 contains four single-qubit gates and two CNOT gates. The NOT gate represented by X in Fig. 2, Eq. (9)] is implemented by a $\pi_x$ pulse. $\theta_1$ and $\theta_2$ [Eq. (9)] are implemented by $(\alpha)_u$ and $(\alpha)_d$ pulses, respectively. The CNOT gate [Eq. (10)] is implemented by the pulse sequence $(\pi/2)_{\gamma}^i-(\pi/2)_{\gamma}^f-(1/2J)_{z}-(\pi/2)_{x}^f-(\pi/2)_{z}^f$, where the $(\pi/2)_{\gamma}^i$ pulse is obtained by the composite pulse $(\pi/2)_{\gamma}^i-(\pi/2)_{\gamma}^f-(\pi/2)_{\gamma}^f$ and the $(\pi/2)_{x}^f$ pulse is obtained by the composite pulse $(\pi/2)_{x}^f-(\pi/2)_{x}^f-(\pi/2)_{x}^f$. The first $(\pi/2)_{\gamma}^i$ pulse of the composite pulse is canceled with the last $(\pi/2)_{\gamma}^f$ pulse of the CNOT gate. All the pulses are applied at resonance, such that the chemical shifts are refocused throughout the pulse sequence.

by using the pulse sequence $(\pi/2)_{\gamma}^i-(\pi/2)_{\gamma}^f-(1/2J)_{z}-(\pi/2)_{x}^f-(\pi/2)_{z}^f$ [46], where the superscript 1 stands for proton and 2 stands for carbon. The $(\pi/2)_{\gamma}^i$ is obtained by the composite pulse $(\pi/2)_{\gamma}^i-(\pi/2)_{\gamma}^f-(\pi/2)_{\gamma}^f$ as shown in Fig. 5. The $(\pi/2)_{\gamma}^i$ pulse can be obtained by another composite pulse $(\pi/2)_{\gamma}^i-(\pi/2)_{\gamma}^f-(\pi/2)_{\gamma}^f$, so that the first $(\pi/2)_{\gamma}^i$ pulse of the composite pulse cancels the last $(\pi/2)_{\gamma}^f$ pulse of the CNOT gate, yielding the last two pulses in the CNOT sequence as $(\pi/2)_{\gamma}^f-(\pi/2)_{\gamma}^f$. All the pulses in the pulse sequence are applied at resonance, so the chemical shifts are refocused throughout the pulse sequence. Hence during the time period $(1/2J)$, the system evolves only under the $J$-coupling Hamiltonian $H_{J}=2\pi J I_{x}I_{y}$, yielding the unitary operator $e^{i\pi I_{x}I_{y}}$.

(iii) The controlled-$\sigma_z$ gate ($\sigma_3^c$) is implemented by a $(\pi/2)^{1,2}$ pulse followed by an evolution for the time $1/2J$ [18]. $(\pi/2)^{1,2}$ pulses are realized by composite rotation on both qubits as shown in Fig. 5.

A. Linearly polarized states

We have studied the linearly polarized case by varying both the parameters $\alpha$ [rotation angle of $U$, Eq. (8)] and $2\theta_i$ [angle between $|\psi_i\rangle$ and $|\psi_f\rangle$, Fig. 1(b)]. The pulse sequence given in Fig. 5 is implemented with the initial state prepared as described above [in (i)]. An experiment is performed to discriminate several pairs of linearly polarized states for $\alpha=30^\circ, 45^\circ, 60^\circ$, and 90°. For each value of $\alpha$, the experiment is carried out for $2\theta_1=20^\circ, 40^\circ, 60^\circ, 80^\circ, 90^\circ, 100^\circ, 120^\circ, 140^\circ$, and 160°. As mentioned in theory section (Sec. II), the experiment is performed twice, one with $\sigma_3^c$ and other without $\sigma_3^c$, and the results are added so that the resultant intensity of the data qubit transition gives the probability of discrimination ($P=2\theta_1^2\sigma_3^2$). Figure 6 contains typical spectra for $2\theta_1=90^\circ$ and $\alpha=90^\circ, 60^\circ, 45^\circ$, and 30°, where the data qubit

![FIG. 5. The pulse sequence for implementation of the quantum circuit of Fig. 2.](image1)

![FIG. 6. Proton spectra of $^{13}$CHCl$_3$ obtained after the implementation of a pulse sequence given in Fig. (5), where the initial states of the data and ancilla qubits are prepared in linearly polarized states [Sec. III, (i)]: (a) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=90^\circ$. (b) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=60^\circ$. (c) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=45^\circ$. (d) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=30^\circ$. (e) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=30^\circ$. (f) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=60^\circ$. (g) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=45^\circ$, and (h) $|\psi_i\rangle=|\psi_i\rangle$, $2\theta_1=90^\circ$, and $\alpha=30^\circ$. In each of the above experiments $|\psi_i\rangle$ is initialized by choosing $2\theta_i$ to satisfy Eq. (12). A complete set of these experiments has been carried out for different values of $\alpha$ by varying $2\theta_1$ and $2\theta_2$ [satisfying Eq. (12)]. The results are plotted in Figs. 7 and 8.)](image2)
is prepared, respectively, in states $|\psi_1\rangle$ [Figs. 6(a)–6(d)] and $|\psi_2\rangle$ [Figs. 6(e)–6(h)]. As shown in Fig. 6, the positive intensities of the resultant peaks indicate that the initial state of data qubit is $|\psi_1\rangle$ and the negative intensities of the resultant peaks indicate that the initial state of data qubit is $|\psi_2\rangle$. The intensity of the peak yields the probability ($P=2\theta_1^2\alpha^2$). In Fig. 6 one can observe that the intensity of the resultant peak (probability of discrimination) changes with $\alpha$. For different values of $\alpha$, the probability of discrimination, $P$ (experimental and simulation results), as a function of $2\theta_1$ is given in Fig. 7. From Fig. 7 one can find the optimum angle $2\theta_1$ for the maximum probability of discrimination for a given value of $\alpha$. Figure 8, on the other hand, shows the variation of probability of discrimination ($P$) as a function of $\alpha$ for different $2\theta_1$. From Fig. 8, one can find the value of $\alpha$ to get the maximum probability of discrimination for a given angle ($2\theta_1$) between $|\psi_1\rangle$ and $|\psi_2\rangle$. In both Figs. 7 and 8, the experimental points agree well with the simulations, confirming successful discrimination of linearly polarized states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the data qubit.

B. Elliptically polarized states

We also discriminate several pairs of elliptically polarized states. Experiments have been performed, using the pulse sequence given in Fig. 5, for $\alpha=90^\circ$ and ellipticities $\epsilon=0^\circ$, $15^\circ$, and $30^\circ$. For each value of $\epsilon$, we perform the experiment for $2\theta_1=20^\circ$, $40^\circ$, $60^\circ$, $90^\circ$, $120^\circ$, $140^\circ$, and $160^\circ$. As de-

FIG. 7. Probability of discrimination $P$ (resultant intensity of the transition of the data qubit $^1$H) from Fig. 6 and corresponding experiments for various $2\theta_1$ and $\alpha$. The $2\theta_2$ is adjusted to satisfy Eq. (12) in each case. The expected intensities (shown by thick lines) are obtained by simulation of the pulse program of the pulse sequence given in Fig. 5 using the MATLAB program. Since the total pulse sequence lasts for about 11.8 ms, which is much less than $T_1$ and $T_2$ of both $^1$H and $^{13}$C, the relaxation effects were not included in the simulation. However, all the experimental data points are normalized with respect to the experimental spectrum of $\alpha=90^\circ$ and $2\theta_1=90^\circ$ for which the intensity is taken as 0.5, which is the theoretical expected intensity. The maximum probability of discrimination ($P_{\text{max}}$) is obtained for $2\theta_1=90^\circ$ for all values of $\alpha$. However, the value of $P_{\text{max}}$ depends on the value of $\alpha$.

FIG. 8. The results shown in Fig. 7 are replotted as a function of $\alpha$ for various $2\theta_1$. The solid curves are simulated results and the experimental data points are shown by crosses. From these curves one can find the optimum value of $\alpha$ for a given $2\theta_1$. 

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IV. CONCLUSION

The implementation of a programmable quantum-state discriminator by NMR has been demonstrated. The device discriminates a pair of data qubit states unambiguously (error free) that are symmetrically located around some fixed state. One can use the same device (without changing its parameters) to discriminate any pair of data qubit states by suitably preparing the ancilla qubit. However, the probability of discrimination depends on the parameter of the device (angle \(\alpha\)). It may be noted that since NMR is an ensemble measurement, it is inevitable that to do projective measurement one has to prepare the input state twice. The probability of successful discrimination is obtained as a function of the angle between pair of data qubit states and the rotation angle of the unitary operator of the protocol. The states of the ancilla (program) qubit that represent different programs can be nonorthogonal, which indicates the quantum nature of the programming. It is further shown that if the pair of data qubits are in elliptically polarized states, then the probability of successful discrimination is also a function of ellipticity.

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APPENDIX

The unitary operator corresponding to a radio-frequency (rf) pulse of angle \(\alpha\) and phase (direction of rf pulse) \(\phi\) is \(R_\phi(\alpha)\), which is also called the \((\alpha, \phi)\) pulse,

\[
R_\phi(\alpha) = e^{-i\pi\hat{n}\cdot \hat{I}},
\]

where \(\hat{n}\) is a unit vector whose direction is along the direction of rf pulse and \(I=I_x\hat{i} + I_y\hat{j} + I_z\hat{k}\), where \(I\) is the angular momentum operator of spin-1/2 nuclei.

In spherical polar coordinates \(\hat{n}=n_x\hat{i} + n_y\hat{j} + n_z\hat{k}\), where \(n\) is the angle between \(\hat{n}\) and \(\hat{z}\) axis (direction of static magnetic field) and \(\phi\) is the angle between \(\hat{n}\) and the \(x\) axis. Here \(\theta=90^\circ\), since the rf pulse is applied perpendicular to the static magnetic field.

After simplification, the unitary operator of the \((\alpha, \phi)\) pulse, \(R_\phi(\alpha)\), can be written as
\( R_{\phi}(\alpha) = \begin{pmatrix} \cos(\alpha/2) & -e^{-i\phi_1} \sin(\alpha/2) \\ e^{i\phi_1} \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}, \)

where \( \phi_1 = \phi - \pi/2 \). Here \( \phi = 0^\circ \) gives the \((\alpha)\) pulse, and \( \phi = 180^\circ \) gives the \((\alpha),_+\) pulse. Similarly \( \phi = 90^\circ \) and \( \phi = 270^\circ \) give \((\alpha),_1\) and \((\alpha),_-\) pulses, respectively. From \( R_{\phi}(\alpha)\), one can calculate any unitary operator, corresponding to any arbitrary angle and phase. For example, the unitary operator corresponding to the \((2\eta),_{(\pi/2+\phi)}\) pulse is

\[
(2\eta),_{(\pi/2+\phi)} = \begin{pmatrix} \cos(\eta) & -e^{-i\phi} \sin(\eta) \\ e^{i\phi} \sin(\eta) & \cos(\eta) \end{pmatrix}.
\]

The unitary operator of the \((2\eta),_{-((\pi/2+\phi)}\) pulse is given by the Hermitian conjugate of the above.